About Borel and almost Borel embeddings for $\mathbb{Z}^d$ actions

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In this talk we will report results with Tom Meyerovitch (2020), ongoing work with Spencer Unger and some open questions.

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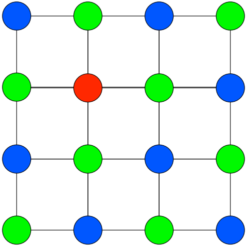
We want to understand the assumptions on the dynamical system (X, T) which implies that it is 'universal'.

By 'universal' we mean that 'any' free system (Y, S) (with low enough entropy) can be Borel embedded into (X, T).

In this talk we focus on colouring of actions as an example.

Chromatic number

The chromatic number of a graph is the minimum number of colours required to properly colour the graph.



The chromatic number of $\mathbb{Z}^d$ is 2.

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In other words in what is the minimum k such that we can partition $X := \sqcup_{i=1}^{k} X_i$ into Borel sets such that if $x \in X_j$ then the neighbours of x are in $\cup_{i \neq j} X_i$.

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This would mean that if μ is an invariant measure for the action then $\mu(X_1) = \mu(X_2) = 1/2$ and both X_1 and X_2 are invariant under T^2. Hence T^2 is not ergodic.

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Gao and Jackson (2015) showed that it is between 2 and 4.

Theorem (Chandgotia & Unger, and by Gao, Jackson, Krohne & Seward)

The chromatic number of a free $\mathbb{Z}^d$ action on a Polish space is either 2 *or* 3.

We will now see a sketch of the proof.

We start with a theorem by Rokhlin.

Theorem (Rokhlin 1948 for $d = 1$ / Katznelson & Weiss 1972 for $d > 1$)

Let (X, T) be a free $\mathbb{Z}^d$ action and $\epsilon > 0$ and $n \in \mathbb{N}$. Then there exists $A \subset X$ such that

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How can we properly colour the space using this?

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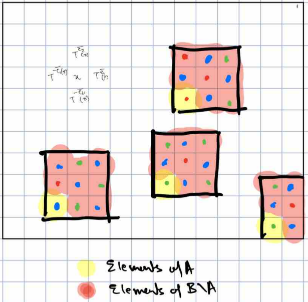
$$\bigsqcup_{\vec{e}\in[1,n]^d} T^{\vec{e}}(A) = B$$

$\forall$ μ a.e. $x \in X$,

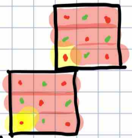
Orbit of x.



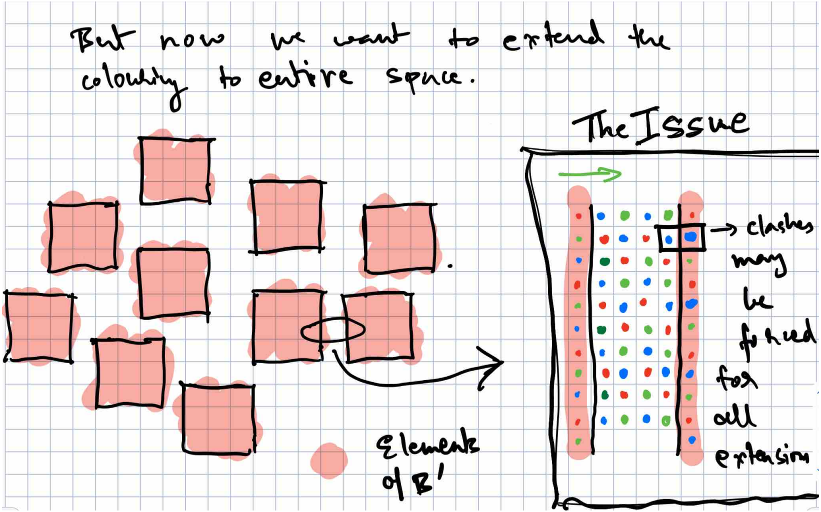
B covers about $(1-\varepsilon)$ proportion of orbit.

Now each element of B can be assigned a colour.

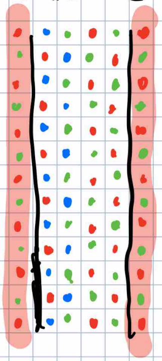
Some of these colours may clash on the boundary.



So we choose $B' = \bigsqcup_{\vec{e}\in[3,n-2]^d} T^{\vec{e}}(A)$

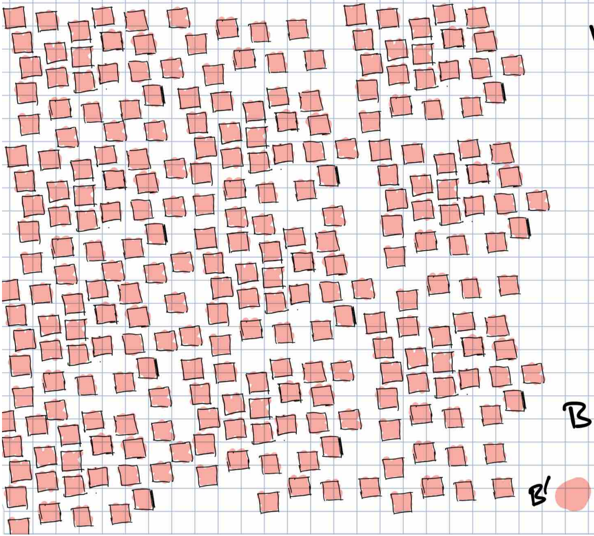

But now we want to extend the colouring to entire space.
The Issue
clashes may be forced for all extension
Elements of B'

We need to colour the boundary of the boxes carefully.-



Checkerboard patterns on the boundary always extend.

So now we have a colouring of B'



We now choose

$n_1 >>> n$

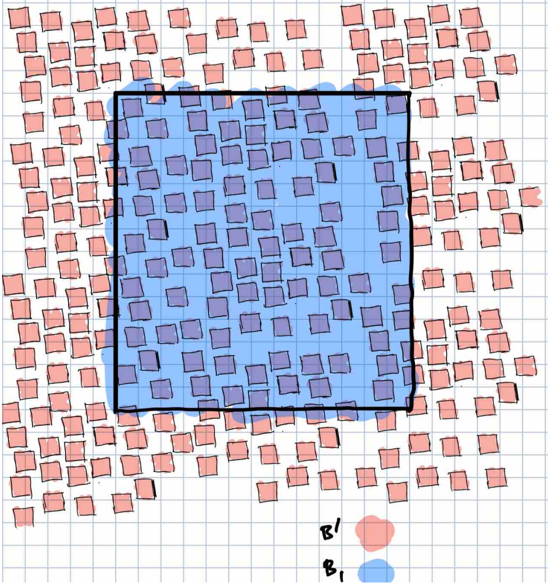
$\varepsilon_1 <<< \varepsilon$.

and find

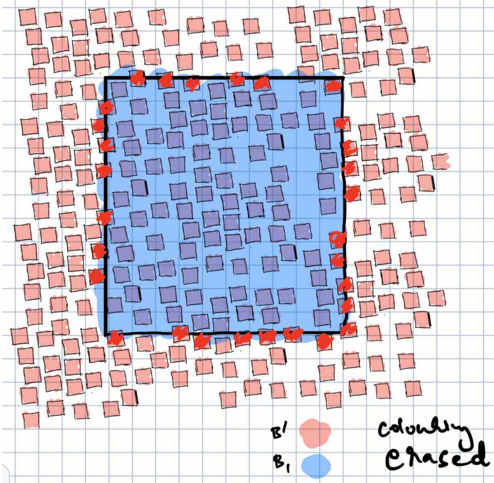
A_1 st

$\mu\left(\bigsqcup_{\bar{e}\in[1,n_1]^d} T^{\bar{e}} A_1\right) > 1-\varepsilon_1$

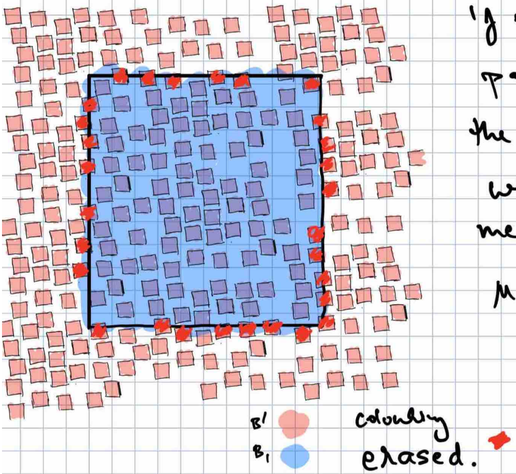
$$B_1 := \bigsqcup_{\bar{e}\in[3,n_1-2]^d} T^{\bar{e}}(A_1)$$



We want to extend the colouring to B_1. So we erase the colours on B' close to boundary



Erase the colouring on the boundary. And now extend the remaining colouring to B_1. Continue in this manner.



If we choose the parameters carefully the erased parts will have small measure, say

$$\mu\left(\begin{array}{l}\text{erased colouring}\\ \text{at step } i\end{array}\right) = \varepsilon_i$$

If $\sum \varepsilon_i < \infty$ by Borel-Cantelli the colouring will be erased only finitely many times μ-a.s.

Şahin-Robinson's colouring

Given a $\mathbb{Z}^d$ action (X, T), a set is called a full set if $\mu(X') = 1$ for invariant probability measures μ.

Theorem (Şahin-Robinson, 2004)

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The answer is yes.

Given a $\mathbb{Z}^d$ action (X, T), by its entropy, we mean the Gurevic entropy, that is, the supermum of the measure theoretic entropy on the space.

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You can assume that it is some measure of size / complexity of the action (X, T).

Theorem (Chandgotia, Meyerovitch 2021)

Let (X, T) *be a free* $\mathbb{Z}^d$ *action of entropy less than the space of 3-colourings. There exists a full set* $X' \subset X$ *which can be embedded into the space of proper 3-colourings in an equivariant manner.*

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Question

Prove that the space of proper 3-colourings is universal, that is, there is no need to get rid of null set to obtain an embedding.

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Our results built upon techniques developed by Hochman (2013), who proved the same result for the full shift in one dimension (strengthening the result for Krieger-1972).

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Şahin and Robinson (2002) proved universality for certain systems assuming certain mixing conditions (which is not satisfied by the systems given above).

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For the space of domino tilings we needed an estimate which was known in $d = 2$ due to Kastelyn (1968) and was recently proved by me for $d > 2$.

What combinatorial estimate do we need?- A major open question

Consider a set of rectangles $T_1, T_2, \ldots, T_p$ such that

$$\gcd(\text{dimension of } T_i \text{ in the } k\text{th direction}; 1 \le i \le p) = 1 \text{ for all } k.$$

Let N be the product of the side lengths. We need to compare perfect tilings of a Nk-box and tilings without any boundary restriction.

Question

Prove that

$$\lim_{k \to \infty} \frac{\log \#(\textit{perfect tilings of a } [1, Nk]^d)}{\log \#(\textit{ tilings of a } [1, Nk]^d)} = 1.$$

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4. (X, T) is space with non-uniform specification.

With the last item we were answering a question by Quas and Soo (2012) who proved this with some additional hypothesis.

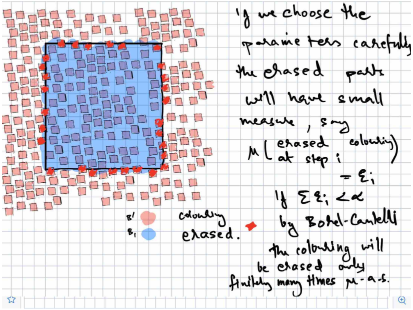
A nice corollary of our work is the following:

Theorem (Chandgotia, Meyerovitch 2021)

A generic homeomorphism (with respect to the sup-metric) of any manifold of dimension > 1 *is almost universal.*

We believe that adjective almost is unnecessary.

What problems are encountered getting rid of the ‘almost’?



We needed that all most every point of the space X belongs to at most finitely many boundaries of Rokhlin towers.

This no longer holds in the Polish setting.

Why can't we use Rokhlin towers directly?

Theorem (Gao, Jackson and Krohne, 2015)

Let $d \geq 2$ and (X, T) be a $\mathbb{Z}^d$ minimal dynamical system such that the subsystem with respect to $\mathbb{Z} \times \{0\}^{d-1}$ is also minimal.

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1. B_n *is a complete section.*
2. *The connected components of* B_n *are finite rectangles growing in size.*

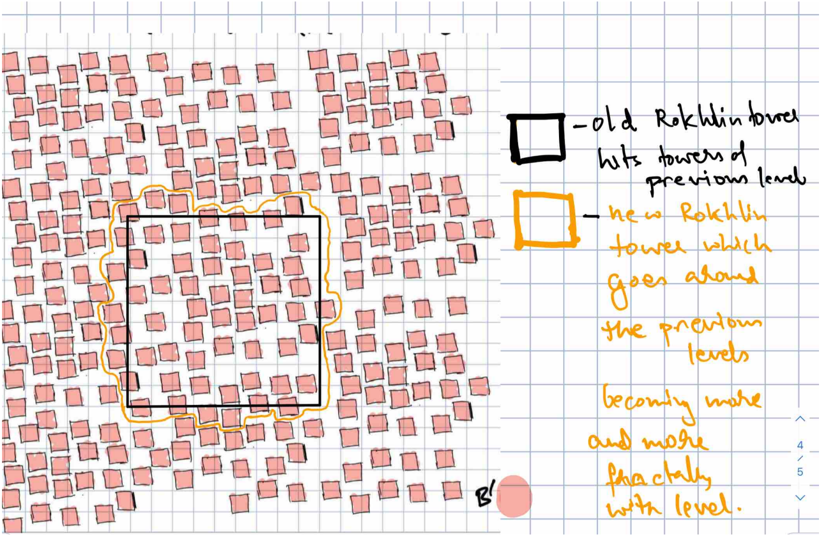
Then the set

$$\{x \in X \ : \ x \in \partial B_n \text{ for infinitely many } n\}$$

is comeager.

Gao, Jackson and Seward also suggested a workaround. A proof of this can be found in a paper by Marks and Unger.

Gao, Jackson and Seward's walkaround



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Again our methods are general enough to show that we can find a factor from any free Polish $\mathbb{Z}^d$ action (X, T) to:

1. The space of tilings by rectangles (under some natural necessary conditions).
2. The space of directed bi-infinite Hamiltonian paths.

The first result extends results of Gao and Jackson who need additional assumptions on the rectangles. It answers question raised by Gao, Jackson and Seward.

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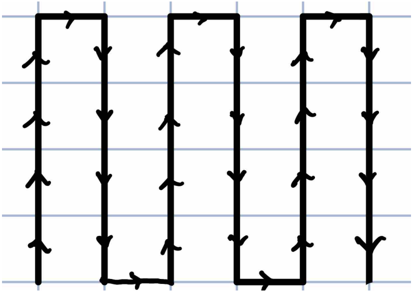
The second result recovers a result announced by Gao, Jackson, Krohne & Seward. Under presence of an ergodic measure this result was announced by Downarowicz, Oprocha & Zhang.

What is a bi-infinite Hamiltonian path?

A directed Hamiltonian path on a graph is a walk on a graph such that every vertex is visited exactly once



This shows that any $\mathbb{Z}^d$ action on a Polish space (X, T) is orbit equivalent to a $\mathbb{Z}$ action (X, S) such that if $S(x) = T^{\vec{e}}(x)$ for some unit vector $\vec{e}$ (depends on x).

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Theorem (Tserunyan 2015)

Let (X, T) be the action of a countable group with no invariant probability measures then it can be embedded in the full shift over 32 *symbols.*

Theorem (Hochman 2019)

Let (X, T) be the action of $\mathbb{Z}$ with no invariant probability measures then it can be embedded in any shift of finite type.

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Again this can be generalised to tilings, the space of bi-infinite Hamiltonian paths.

Our methods are completely different from the previous proofs of such results but restricted to symbolic spaces.

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Theorem (Gao & Jackson 2015)

A continuous 3*-colouring of the free part of the* 2*-full shift does not exist (but a* 4*-colouring does).*

Theorem (Salo, 2021)

There is no continuous embedding of the space of proper 3 *colourings into the* 2 *full shift.*

Open questions

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3. Let $\mathbb{T}$ be a set of coprime boxes. Let N be the product of length of the sides of $\mathbb{T}$. Prove that

$$\lim_{n\to\infty} \frac{1}{N^d n^d} \log(\text{the number of tilings of } [1, Nn]^d \text{ by elements of } \mathbb{T}) = \text{ topological entropy of all the tilings of } \mathbb{T}.$$

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